# Cantor and sets: La diagonale du fou 

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## Cantor and sets: La diagonale du fou

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## 1 Cantor's paradise

### 1.1 Introduction

- Before Cantor: no sets in maths!
- Cantor studied trigonometric series: if for all real $x$

$$
\sum_{n=0}^{N} a_{n} \sin (n x)+b_{n} \cos (n x) \xrightarrow[N \rightarrow+\infty]{ } 0
$$

then for $n, a_{n}=b_{n}=0$.

- What if the series does converge for all $x$ but a few exceptions?


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- What if there is an infinity of exceptions?
$\rightarrow$ study of the set of reals where the series does not converge.
- Is the set finite?
- Is it possible to index it by natural numbers?


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### 1.2 Countable sets

A set $E$ is denumerable if and only if there exists a bijective mapping from $\mathbf{N}$ to this set.

A set is countable if it is denumerable or finite.

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## Exemple:

1. $\mathbf{N}$ is denumerable
2. Z too
3. An denumerable union of finite sets is countable
4. A subset of a denumerable set is countable
5. $\mathbf{N} \times \mathbf{N}$ too
6. Q too

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### 1.3 R is not countable

Diagonal argument (1874/1891): given an infinite sequence of reals, you can build a real which is not in the sequence.

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The idea: the diagonal argument

$$
\begin{aligned}
& u_{1}=1 . \underline{2} 983379557834 \ldots \\
& u_{2}=7.7 \underline{9} 52039204368 \ldots \\
& u_{3}=0.82 \underline{0} 4928239869 \ldots \\
& u_{4}=0.027 \underline{5} 923093482 \ldots \\
& u_{5}=0.0568 \underline{7} 39503894 \ldots
\end{aligned}
$$

Construct $x$ such that the $k$ th digit of $x$ differs from the $k$ th of $u_{k}$. For instance $x=0.53121 \ldots$

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## Consequences:

- $\mathbf{R}$ is not countable
- not all infinite sets have the same "number" of elements


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### 1.4 Comparing sets

Given two sets $A$ and $B$, we say:

- they are equinumerous if and only if there exists a bijection from $A$ to $B$. In that case we also say that $A$ and $B$ have the same cardinality and we write $|A|=|B|$.
- A has cardinality smaller than or equal to the cardinality of $B$ if and only if there exists an injective function from $A$ to $B$ and we write $|A| \leq|B|$.

With this definition

$$
|A|=|B| \Rightarrow|A| \leq|B| \text { and }|B| \leq|A|
$$

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## Theorem (Cantor-Bernstein)

$$
\left.\begin{array}{rl}
|A| & \leq|B| \\
\text { and }|B| & \leq|A|
\end{array}\right\} \Rightarrow|A|=|B|
$$

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### 1.5 Cardinals

Notation $|A|=|B|$ : more than just a notation for equinumerosity.
$|A|$ is a set, called the cardinal of $A$ such that:

1. $|A|$ and $A$ are equinumerous
2. equinumerosity of sets $\Longleftrightarrow$ equality of their cardinals
3. $|A| \leq|B| \Longleftrightarrow|A| \subset|B|$.

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## Moreover:

1. $|\emptyset|=\emptyset$, notation: 0 .
2. $\forall x|\{x\}|=\{0\}$, notation: 1 .
3. $\forall x, y x \neq y \Rightarrow|\{x, y\}|=\{0,1\}$, notation: 2 . 4. ...

We define $\aleph_{0}=|\mathbf{N}|$.

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## New questions

1. Smaller infinite sets than $\mathbf{N}$ ? No
2. Bigger sets than $\mathbf{R}$ ? Yes
3. Existence of $E$ such that $|\mathbf{N}|<E<|\mathbf{R}|$ ? Undecidable (Continuum problem, Gödel 1940, Cohen 1963)

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## Bigger sets than $R$ ?

Diagonal argument: let $f: \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$.
$f$ can not be a surjective function. Why?
(Hint: define $E=\{x \in \mathbf{R} \mid x \notin f(x)\}$.)

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## Beth vertigo

The previous argument applies to all sets, not only to $\mathbf{R}$.
Notice that $|\mathcal{P}(A)|=\left|\{0,1\}^{A}\right|=\left|2^{\mathfrak{c}}\right|$ where $\mathfrak{c}=|A|$.
Writing $2^{\mathfrak{c}}$ instead of $\left|2^{\mathfrak{c}}\right|$, let us show that $\mathfrak{c}<2^{\mathfrak{c}}$.
Consider $f: \mathfrak{c} \rightarrow\{0,1\}^{\mathfrak{c}}$.
Define $\phi: x \mapsto 1-f(x)(x)$. Then for all $y \in \mathfrak{c}, f(y) \neq \phi$.
We define $\beth_{0}={ }_{\text {def }} \aleph_{0}$ and for all $n \in \mathbf{N}, \beth_{n+1}={ }_{\text {def }} 2^{\beth_{n}}$.

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Computing with cardinals: given cardinals $\mathfrak{c}_{1}, \mathfrak{c}_{2}, \mathfrak{c}_{3}$

$$
\begin{aligned}
\mathfrak{c}_{1}+\mathfrak{c}_{2} & =\text { def }\left|\left\{(i, x) \mid i \in\{0,1\}, x \in \mathfrak{c}_{i}\right\}\right| \\
\mathfrak{c}_{1} \times \mathfrak{c}_{2} & ={ }_{\text {def }}\left|\mathfrak{c}_{1} \times \mathfrak{c}_{2}\right| \\
\mathfrak{c}_{1}^{\mathfrak{c}_{2}} & ={ }_{\text {def }}\left|\mathfrak{c}_{1}^{\mathfrak{c}_{2}}\right|
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathfrak{c}_{1}^{\mathfrak{c}_{3}} \times \mathfrak{c}_{2}^{\mathfrak{c}_{3}} & =\left(\mathfrak{c}_{1} \times \mathfrak{c}_{2}\right)^{\mathfrak{c}_{3}} \\
\mathfrak{c}_{1}^{\mathfrak{c}_{2}} \times \mathfrak{c}_{1}^{\mathfrak{c}_{3}} & =\mathfrak{c}_{1}^{\mathfrak{c}_{2}+\mathfrak{c}_{3}} \\
\left(\mathfrak{c}_{1}^{\mathfrak{c}_{2}}\right)^{\mathfrak{c}_{3}} & =\mathfrak{c}_{1}^{\mathfrak{c}_{2} \times \mathfrak{c}_{3}}
\end{aligned}
$$

(can you prove it?)

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$$
\begin{gathered}
\beth_{0}=\aleph_{0}=|\mathbf{N}|=|\mathbf{Z}|=|\mathbf{Q}| \\
\beth_{1}=|\mathbf{R}|=\left|\mathbf{R}^{2}\right|=\left|\mathbf{R}^{n}\right|=\left|\mathbf{R}^{\mathbf{N}}\right|=\left|\mathbf{R}^{\mathbf{Q}}\right|=\left|\mathcal{C}^{0}(\mathbf{R}, \mathbf{R})\right| \\
\beth_{2}=\left|\{0,1\}^{\mathbf{R}}\right|=\left|\mathbf{N}^{\mathbf{R}}\right|=\left|\mathbf{R}^{\mathbf{R}}\right|
\end{gathered}
$$

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## 2 Paradise or hell?

### 2.1 Russell paradox

1. Mathematics collapse if you find a proof of an assertion and of its negation. Such a proof is called a paradox
2. In May or June 1901 Bertrand Russel discovered such a proof in Cantor's set theory ${ }^{1}$
[^0]
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Considering that

1. the set of integers is not an integer (that is quite normal)
2. the set of all sets, if it existed, would be a member of itself (quite a strange set)

You could divide sets into normal sets and non-normal sets.
Let us call $N$ the set of normal sets, that is

$$
N=\{A \mid A \notin A\}
$$

Is $N$ a normal set itself?

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### 2.2 Repairing set theory

In order to repair set theory, two distinct solutions have been found:

1. Type theory
2. Set theory

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## Type theory (notable proponents: Russel and Whitehead)

1. Mathematical objects all have a level:

- Base objects (integers) live in level 0
- Sets containing objects of level 0 live in level 1
- Sets containing objects of level at most $n$ live in level $n+1$

2. Useful in computer science (origin of the type system of Caml)
3. Inconvenient for mathematics.

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Set theory

1. Rule for forming sets restricted
2. Gave birth to Zermelo-Fraenkel set theory (with the choice axiom). Known as ZFC.
3. You use it everyday

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### 2.3 ZFC

Theory proposed around 1920.
Distinction between predicates and sets:

1. ok to define $P$ such that

$$
\forall x \quad P(x) \Longleftrightarrow x \notin x
$$

2. Forming the set $\{x \mid P(x)\}$ is forbidden.

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Two axioms about sets you can form using a predicate:

Schema of separation if $A$ is already a set, and $P$ any predicate, then you can form

$$
\{x \in A \mid P(x)\}
$$

Schema of replacement if $A$ is a set, and $R$ is a two-arguments predicate such that $\forall x \in A \exists!y R(x, y)$, then you can form

$$
\{y \mid \exists x \in A R(x, y)\}
$$

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### 2.4 Consistency

Is ZFC consistent?

1. Up to 2010, no paradox has been found in ZFC. . .
2. If ZFC is consistent, no proof of consistency in itself can be done.
3. Some results (Werner, 1997) show (roughly) the relative consistency of ZFC with respect to type theory and vice-versa.

## 3 Applications to Computer Science

3.1 Non-computable functions

A (naive) view of computer programs:

1. We consider only programs inputing a natural number and outputting a natural number.
2. So a program $p$ is a function $\mathbf{N} \rightarrow \mathbf{N}$.

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A compiler/an interpreter is a mapping $\phi$ from the set of program sources to $\mathbf{N}^{\mathbf{N}}$.

Program sources are a countable set, so we can see $\phi$ as a function $\mathbf{N} \rightarrow \mathbf{N}^{\mathbf{N}}$.

Uncountable number of functions: $\left|\mathbf{N}^{\mathbf{N}}\right|=\beth_{1}$
Countable number of programs: $|\phi(\mathbf{N})| \leq|\mathbf{N}|=\aleph_{0}$

Thus there are many non-computable functions (an uncountable number), whatever language you use.

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### 3.2 Partial functions

Define $f: n \mapsto 1+\phi(n)(n)$.
Paradox(?):

- On the one hand: In a any reasonably powerful programing language, you can

1. simulate an interpreter of itself
2. add 1 to a previously computed value

Thus, $f$ should be computable.

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- On the other hand:

1. if $f$ is computable, then $f=\phi(N)$ for some $N$.
2. Then $\phi(N)(N)=f(N)=1+\phi(N)(N)$ which is absurd.

What is wrong?
Implicit assumptions:

- the language is powerful enough
- no infinite loops


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These assumptions are mutually incompatible:

## Theorem

In all reasonably powerful programing language, some valid programs loop.

Better description of a program: function $\mathbf{N} \rightarrow \mathbf{N} \cup\{\perp\}$ where $\perp$ means the program loops.

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### 3.3 The halting problem

Infinitely looping programs often are the result of a bug.

Can we automatically detect them before running them?

Halting problem: find a computer program inputing the text of a program and a natural number and telling whether the given program would stop on the given entry.

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Assume that such a program exists: let call $h$ a program inputing the text of a program $x$ and a natural number $y\left((x, y) \in \mathbf{N}^{2}\right)$ and outputting 1 if $\phi(x)(y)$ would stop, and 0 otherwise.

Now define $f$ a the function inputing $x$ and

- outputting 0 if $h(x, x)=0$
- infinitely looping if $h(x, x)=1$

Then, choose $N \in \mathbf{N}$ such that $f=\phi(N)$.
Does $f(N)$ loops?
Conclusion?

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### 3.4 Another related oddity

Can you write a Caml program forever looping

1. without any loop construct
2. without any recursive function
3. in four lines?

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## Answer:

Objective Caml version 3.11.2
\# type $\mathrm{t}=\mathrm{L}$ of ( $\mathrm{t}-\mathrm{t}$ ); ;
type $t=L$ of ( $t->\mathrm{t}$ )
\# let app $x$ y $=$ match $x$ with (L f) $\rightarrow \mathrm{f} y ;$;
val app : t -> t -> t = <fun>
\# let delta $=$ L (fun $\mathrm{x}->\mathrm{app} \mathrm{x}$ x);
val delta : t = L <fun>
\# app delta delta; ;
infinite loop...

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## 4 Applications to mathematical logic

Gödel proved around 1930 that in all logical system, you can find an assertion $A$ such that neither $A$ neither the negation of $A$ can be proved (incompleteness theorem).

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### 4.1 Free variables

Let $A$ be the assertion $\forall y \in \mathbf{N} y \geq x-z$.
The free variables of $A$ are $x$ and $z$ ( $y$ is a bound variable).
The truth value of this assertion (true or false) may depend on the value of its free variables.

An assertion is closed if and only if it has no free variables.

In the following, we shall consider only the case of closed assertions

### 4.2 Truth value / provability

Platonic mathematical view of truth: every closed assertion is either true or false (even if you do not know whether it is true or false). If it is false, its negation is true and vice-versa.

A proof of an assertion is a text following some (well-defined rules).
An assertion is provable if it has a proof.
Consistent system: all provable assertions are true.
An assertion $A$ is decidable if $A$ or its negation is provable.

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Theorem (Gödel incompleteness theorem)
In all consistent systems, there is at least one undecidable closed assertion.

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A possible proof of Gödel's theorem (not the historical one):

1. Assertions and proof texts can be coded in order to be processed by programs.
2. If an assertion $A$ is decidable, one can write a program outputting 1 if it is true and 0 if it is false. Silly algorithm: try all proof texts in sequence (they are denumerable), stop as soon as you find a proof of $A$ (output 1 in that case) or a proof of its negation (output 0).
3. If the assertion « $\phi(x, y)$ stops» were decidable for all $x$ and $y$, the halting problem would be solvable.

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Disturbing consequences:

1. There are true assertions that you can not prove
2. If you want to prove them, you need more axioms
3. Then you have more provable assertions
4. But Gödel's theorem still applies to your new system: Some unprovable true assertions remain!
5. However many axioms you add, true unprovable assertions remain
6. So, there was an infinity of such assertions at the beginning

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## 5 Conclusion

1. Cantor opened a new playground for mathematician. A paradise (according to Hilbert), but also an original way to get insane. . .
2. Set theory changed the face of maths. The vast majority of maths are currently based on set theory.
3. Gödel work changed the way mathematicians see maths, and even the way philosophers see sciences (fall of the positivism).
4. Strong connections between (theoretical) computer science and the discoveries of Cantor.

[^0]:    ${ }^{1}$ Zermelo found it independently around the same years

